A MATHEMATICAL MODEL FOR AN OLIVE TREE

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Summary

Introduction to the problem (1)

- Olive tree (*Olea europaea* L.) has a great importance in the Mediterranean region.
- The main diseases that affect olive trees are mostly caused by fungi and bacteria, which can infect several parts of the plant (roots, stem, fruits and leaves).
- Nowadays, olive diseases control programs rely mostly on chemical control by application of copper-based fungicides.
- In olives production, plant protection strategy must follow the Guidelines for integrated production of olives.
- A need to develop novel and environmental-friendly control strategies for management of olive diseases is an important research topic.

Introduction to the problem (2)

- Phyllosphere-associated microorganisms may be explored for designing new strategies for the biological control of olive diseases.
- The aerial parts of the plants (phyllosphere) are colonized by a diverse microbial community (mostly bacteria and filamentuous fungi), which can grow on the surface of plant tissues.
- Those microorganisms interact with each other and with host plant, mediating several ecosystem processes by altering plant traits including disease resistance traits.
- Phyllosphere microorganisms can reduce the infection of plant tissues by pathogens either directly, through the production of antagonistic molecules and competition for resources, or indirectly by induction of plant resistance response.

The mathematical model: notations

We consider a single olive tree that is affected by a disease caused by fungi. We assume that there is a second type of fungi on the olive tree, that have a positive effect.
The four populations that are modelled in the ecosystem are

- S: the healthy branches and fruits of the olive tree;
- I: the branches and fruits of the same olive tree that are infected by bad fungi.
- F: the pathogenic filamentuous fungi, attacking and infecting the olive branches and fruits.
- G: the phyllosphere microorganisms, they essentially remove the F's, benefiting then by getting more space for their growth and more food directly from the plant; with this behaviour, they benefit the healthy parts of the plant, S.

The mathematical model (1)

The model reads:

$$\begin{aligned}
\frac{dS}{dt} &= s\left(1-\frac{S+I}{K}\right)S-\lambda SF+bGS \\
\frac{dI}{dt} &= \lambda SF-qIF-gI-s\frac{S+I}{K}I \\
\frac{dF}{dt} &= hqIF-aGF-mF-rF^2-gIF_{av}-s\frac{S+I}{K}IF_{av} \\
\frac{dG}{dt} &= ebGS+uaGF-nG-pG^2-gIG_{av}-s\frac{S+I}{K}IG_{av},
\end{aligned} \tag{1}$$

Whit

$$F_{av}=\frac{F}{I} \quad \text{and} \quad G_{av}=\frac{G}{I}.$$

All the parameters of the model are non-negative and we furthermore assume $h, u \in [0,1]$.

The mathematical model (2)

Substituting $F_{av} = FI^{-1}$ and $G_{av} = GI^{-1}$ in the last two equations of model (1), after simplifications the model becomes

$$\begin{aligned}
\frac{dS}{dt} &= s\left(1-\frac{S+I}{K}\right)S - \lambda SF + bGS \qquad (2)\\
\frac{dI}{dt} &= \lambda SF - qIF - gI - s\frac{S+I}{K}I\\
\frac{dF}{dt} &= hqIF - aGF - mF - rF^2 - gF - s\frac{S+I}{K}F\\
\frac{dG}{dt} &= ebGS + uaGF - nG - pG^2 - gG - s\frac{S+I}{K}G
\end{aligned}$$

The equilibrium points and the stability (1)

Solving the system for the equilibrium there are five feasible equilibria under appropriate requests on the parameters values. To study the stability of the equilibria we evaluate the Jacobian matrix at the fixed points founded

- the trivial equilibrium point $E_0(0,0,0,0)$
- the disease-bad fungi-and-good fungi-free point $E_1(S_1,0,0,0)$
- the good fungi-free point $E_2\left(S_2, G_2, F_2, 0\right)$
- the disease-and-bad fungi-free point $E_3\left(S_3, 0, 0, G_3\right)$
- the coexistence point $E^*\left(S^*, I^*, F^*, G^*\right)$

then we compute the characteristic polynomials in order to have the associated eigenvalues.

The equilibrium points and the stability (2)

- The trivial equilibrium point $E_0(0,0,0,0)$ always feasible. E_0 is unstable since one of the eigenvalues is positive: $\mu_1 = s$, $\mu_2 = -g$, $\mu_3 = -g - m$ and $\mu_4 = -g - n$.
- $E_1(K,0,0,0)$ always feasible. If $K < (g+n+s)(eb)^{-1}$ then E_1 is stable.

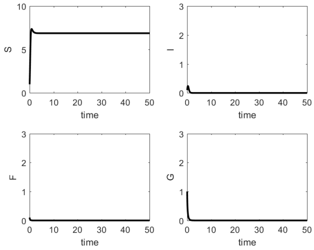


Figure: The equilibrium E_1 at the stability.

The equilibrium points and the stability (3)

- $E_2\left(S_2, \dfrac{m+g+s+(\lambda-r)F_2}{hq}, F_2, 0\right)$ with

$$S_2 = \frac{shqK - (m+g+s)s + \left[s(\lambda-r) - K\lambda hq\right]F_2}{hqs}$$ and

the positive root

$$F_2 = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

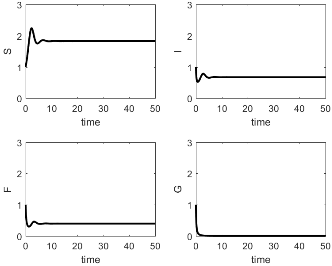
of the quadratic equation $AF^2 + BF + C = 0$, with A, B and C depending on the parameters of the model. The analytical expressions for feasibility of E_2 were found.

The equilibrium points and the stability (4)

For E_2 we have $\mu_1 = uaF_2 + ebS_2 - g - n - sK^{-1}(S_2 + I_2)$, and the other three eigenvalues are the roots of

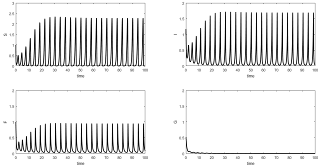
$$\mu^3 + R_1\mu^2 + R_2\mu + R_3 = 0$$

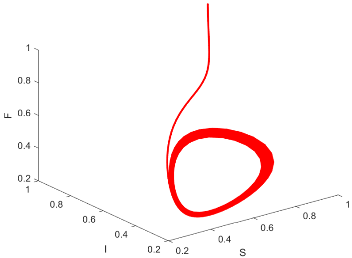
with R_1 is a positive constant and the signs of R_2 and R_3 depend on the parameters values. If $\mu_1 < 0$, $0 < R_3 < R_1 R_2$ then E_2 is stable.



The equilibrium points and the stability (5)

In the next Figures one can see the oscilations of E_2 and the limit cycle in the phase space.





The equilibrium points and the stability (6)

- $E_3\left(\frac{K(bg+bn-ps)}{Kb^2e-bs-ps}, 0, 0, \frac{s(g+n+s-bKe)}{Kb^2e-bs-ps}\right)$ feasible and stable if

$$K < max\left(\frac{g+n+s}{be}, \frac{s(p+b)}{b^2e}\right)$$

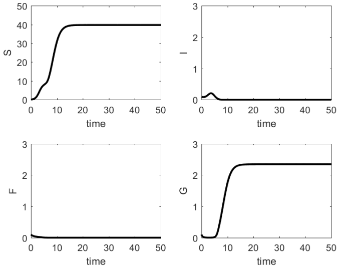


Figure: The equilibrium E_3 at the stability.

The equilibrium points and the stability (7)

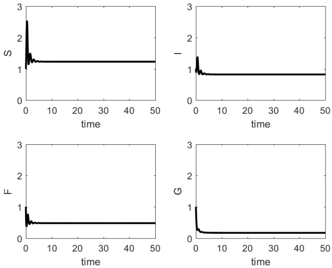
$$E^* \left(S^*, \frac{-A_F F^* - A_C}{A_I}, F^*, \frac{hqI^* + (\lambda - r)F^* - m - g - s}{a+b} \right),$$

with

$$S^* = \frac{sK - sI^* - K\lambda F^* + KbG^*}{s}$$

and F^* the positive root of $A_1 F^2 + B_1 F + C_1 = 0$.

The conditions for feasibility and stability of E^* were found analytically.



Bistability (1)

$E_1 = [6.8, 0.0, 0.0, 0.0]$ and $E^* = [1.2, 0.8, 0.4, 0.1]$ both stable.

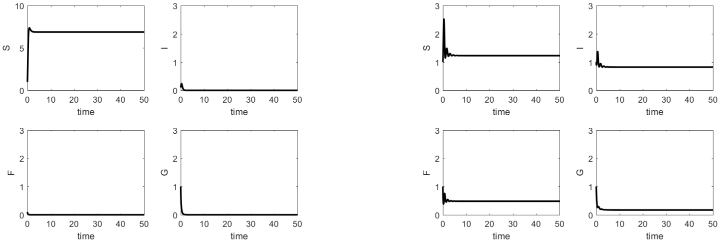


Figure: Left: The equilibrium E_1 at the stability; Right: The equilibrium E^* at the stability.

Bistability (2)

$E_2 = [1.8, 0.6, 0.3, 0.0]$ and $E_3 = [39.7, 0.0, 0.0, 2.3]$ both stable.

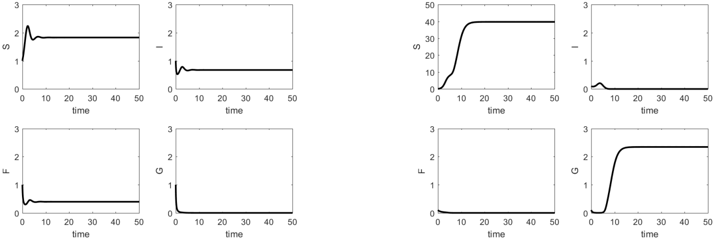


Figure: Left: The equilibrium E_2 at the stability; Right: The equilibrium E_3 at the stability.

Bistability (3)

$E_1 = [6.8, 0.0, 0.0, 0.0]$ and $E_2 = [1.1, 0.6, 0.3, 0.0]$ both stable.

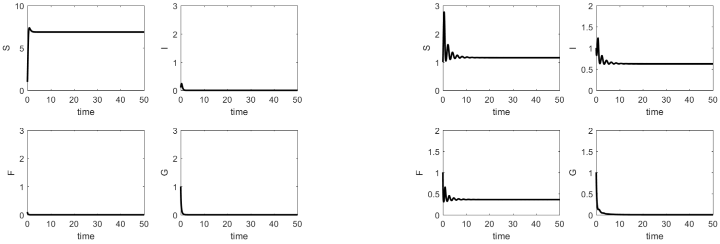


Figure: Left: The equilibrium E_1 at the stability; Right: The equilibrium E_2 at the stability.

Bistability and separatrix surface

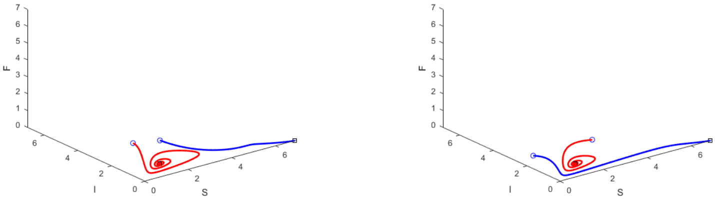


Figure: The phase space representing two different trajectories converging to E_1 and E_2 respectively. Blue dots: the initial conditions; Black squares: the point where the trajectories reach the stability; Blue line: the trajectory that go to E_1; Red line: the trajectory that go to E_2.

Bistability and separatrix surface

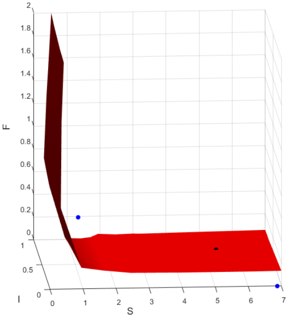


Figure: The basin of attraction of $E_1 = [6.8, 0.0, 0.0, 0.0]$ and $E_2 = [1.1, 0.6, 0.3, 0.0]$.

Bifurcation diagram varying the infection rate

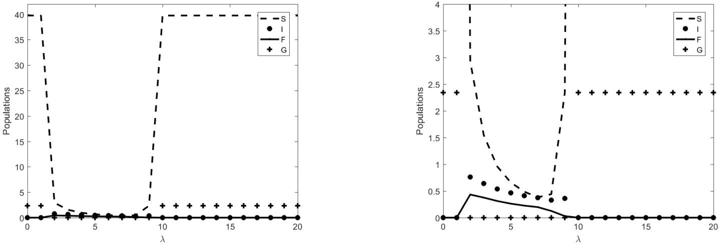


Figure: The transcritical bifurcation from the disease-and-bad fungi-free point, E_3, to the good fungi-free point, E_2, and finally E_3, varying $\lambda \in [0, 30]$. Right: the zoomed version.

Conclusions (1)

- a mathematical model for an olive tree whit two different microorganisms is introduced
- the qualitative analysis and the numerical simulations whit Matlab were done
- the model has five equilibrium point
- from a biological point of view the disease-and-bad fungi-free point, E_3, is the most important, as well as the disease-bad fungi-and-good fungi-free point, E_1
- E_3 is feasible and stable if the carrying capacity of the olive tree, K, is less than a certain threshold

Conclusions (2)

- we have the bistability of three pairs of equilibria
- the most important pairs of stable equilibria are (E_3, E_2) and (E_3, E_1) (E_2 the good fungi-free point)
- a way to chose the right initial conditions in practice could be made using pruning, thus reducing the quantity of infected parts and of bad fungi respectively, and/or by adding more good fungi to the olive tree
- further information arises from the bifurcation diagram varying the infection rate λ, if this value is under a certain threshold or greater that a second threshold then the disease-bad fungi-and-good fungi-free point is stable

Bibliography

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Thanks for your attention!